A characterization of b-chromatic and partial Grundy numbers by induced subgraphs

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Abstract

Gyárfás et al. and Zaker have proven that the Grundy number of a graph G satisfies $\Gamma(G) \geq t$ if and only if G contains an induced subgraph called a t-atom. The family of t-atoms has bounded order and contains a finite number of graphs. In this article, we introduce equivalents of t-atoms for b-coloring and partial Grundy coloring. This concept is used to prove that determining if $\varphi(G) \geq t$ and $\partial\Gamma(G) \geq t$ (under conditions for the b-coloring), for a graph G, is in XP with parameter t. We illustrate the utility of the concept of t-atoms by giving results on b-critical vertices and edges, on b-perfect graphs and on graphs of girth at least 7.

1 Introduction

Given a graph G, a *proper k-coloring* of G is a surjective function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E(G)$; the *color class* V_i is the set $\{u \in V | c(u) = i\}$ and a vertex v has *color i* if $v \in V_i$. We denote by $N(u)$ the set of neighbors of a vertex u and by $N[u]$ the set $N(u) \cup \{u\}$. A vertex v of color i is a *Grundy vertex* if it is adjacent to at least one vertex colored j, for every $j < i$. A *Grundy k-coloring* is a proper k-coloring such that every vertex is a Grundy vertex. The *Grundy number* of a graph G, denoted by $\Gamma(G)$, is the largest integer k such that there exists a Grundy k-coloring of G [10]. A *partial Grundy k-coloring* is a proper k-coloring such that every color class contains at least one Grundy vertex. The *partial Grundy number* of a graph G, denoted by $\partial\Gamma(G)$, is the largest integer k such that there exists a partial Grundy k-coloring of G. Let G and G' be two graphs. By $G \cup G'$ we denote the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. Let $m(G)$ be the largest integer

Figure 1: The graph $K_{3,3}^-$ with $\varphi(K_{3,3}^-) = 2$ (on the left) and $\varphi_r(K_{3,3}^-) = 3$ (on the right).

m such that G has at least m vertices of degree at least $m-1$. A graph G is *tight* if it has exactly $m(G)$ vertices of degree $m(G)-1$.

Another coloring parameter with domination constraints on the colors is the *b-chromatic number*. In a proper-k-coloring, a vertex v of color i is a *b-vertex* if v is adjacent to at least one vertex colored j, $1 \leq j \neq i \leq k$. A *b-k-coloring*, also called b-coloring when k is not specified, is a proper k-coloring such that every color class contains at least one b-vertex. The *b-chromatic number* of a graph G, denoted by $\varphi(G)$, is the largest integer k such that there exists a b-k-coloring of G. In this paper, we introduce the concept of *b-relaxed number*, denoted by $\varphi_r(G)$. A *b-k-relaxed coloring* of G is a b-k-coloring of a subgraph of G. The b-relaxed number of G is $\varphi_r(G) = \max_{H \subseteq G}(\varphi(H))$, for H an induced subgraph of G. Note that we have $\varphi(G) \leq \varphi_r(G) \leq \partial\Gamma(G)$. The difference between $\varphi(G)$ and $\varphi_r(G)$ can be arbitrary large. Let $K_{n,n}^-$ denotes the complete bipartite graph $K_{n,n}$ in which we remove $n-1$ pairwise non incident edges (or $n-1$ edges of a perfect matching in $K_{n,n}$) [1]. For this graph we have $\varphi(K_{n,n}^-) = 2$ and $\varphi_r(K_{n,n}^-) = n$ as Figure 1 illustrates it (for $n = 3$).

The concept of b-coloring has been introduced by Irving and Manlove [16], and a large number of papers was published (see e.g. [8, 19]). The b-chromatic number of regular graphs has been investigated in a serie of papers ([6, 17, 20, 22]). Determining the b-chromatic number of a tight graph is NP-hard even for a connected bipartite graph [18] and a tight chordal graph [12].

In this paper, we study the decision problems b-COL, b-r-COL and pG-COL with parameter t from Table 1.

	b-COL	b-r-COL	G-COL	pG-COL
Question	Does $\varphi(G) \geq t$?	Does $\varphi_r(G) \geq t$?	Does $\Gamma(G) \geq t$?	Does $\partial\Gamma(G) \geq t$?
Complexity class	undetermined	XP	XP [23]	XP

Table 1: The different decision problems with input a graph G and parameter t and their complexity class.

A decision problem is in FPT with parameter t if there exists an algorithm which resolves the problem in time $O(f(t)\ n^c)$, for an instance of size n, a computable function f and a constant c. A decision problem is in XP with parameter t if there exists an algorithm which resolves the problem in time $O(f(t)\ n^{g(t)})$, for an instance of size n and two computable functions f and g.

The concept of t-atom was introduced independently by Gyárfás et al. [11] and by Zaker [23]. The family of t-atoms is finite and the presence of a t-atom can be determined in polynomial time for a fixed t. The following definition is slightly different from the definitions of Gyárfás et al. or Zaker, insisting more on the construction of every t-atom (some t-atoms can not be obtained with the initial construction of Zaker).

Definition 1.1 ([23]). *The family of t-atoms is denoted by $\mathcal{A}_t^\Gamma$, for $t \geq 1$, and is defined by induction. The family $\mathcal{A}_1^\Gamma$ only contains K_1. A graph G is in $\mathcal{A}_{t+1}^\Gamma$ if there exists a graph G' in $\mathcal{A}_t^\Gamma$ and an integer m, $m \leq |V(G')|$, such that G is composed of G' and an independent set I_m of order m, adding edges between G' and I_m such that every vertex in G' is adjacent to at least one vertex in I_m.*

Moreover, in the following sections, we say that a graph G in a family of graphs $\mathcal{F}$ is *minimal*, if no graphs of $\mathcal{F}$ is a proper induced subgraph of G. For example, a minimal t-atom A is a t-atom for which there are no t-atoms which are induced in A other than itself.

Theorem 1.1 ([11, 23]). *A graph G satisfies $\Gamma(G) \geq t$ if and only if it contains an induced minimal t-atom.*

In this paper we prove equivalent theorems for b-relaxed number and partial Grundy number. In contrast with the minimal t-atoms, we can not define the minimal t-atoms for b-coloring as the smallest graphs such that G satisfies $\varphi(G) = t$ (also called b-critical graphs).

The paper is organized as follows: Section 2 is devoted to the definition of t-atoms for the partial Grundy coloring. This concept allows us to prove that the partial Grundy coloring problem is in XP with parameter t. Section 3 is similar to Section 2 but for b-relaxed-coloring. Section 4 is devoted to the concept of b-critical vertices and edges. Section 5 is about b-perfect graphs. Finally, Section 6 deals with graphs for which the b-relaxed and the b-chromatic numbers are equal.

2 Partial-Grundy-t-atoms: t-atoms for partial Grundy coloring

We start this section with the definition of t-atoms for partial Grundy coloring.

Definition 2.1. *Given an integer t, a partial Grundy t-atom (or pG-t-atom, for short) is a graph A whose vertex-set can be partitioned into t sets $D_1, \ldots, D_t$, where D_i contains a special vertex c_i for each $i \in \{1, \ldots, t\}$ such that the following holds:*

- *For all $i \in \{1, \ldots, t\}$, D_i is an independent set and $|D_i| \leq t - i + 1$;*

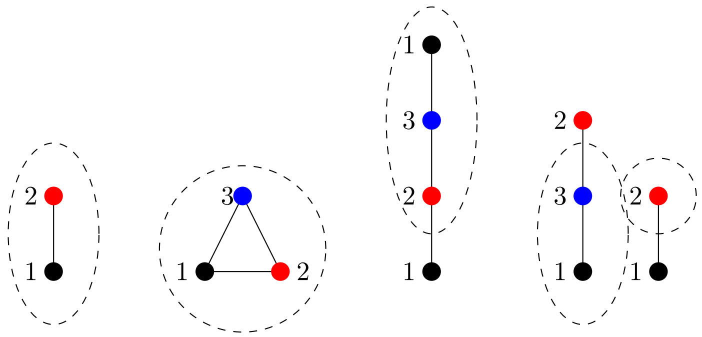

Figure 2: The minimal pG-2-atom (on the left) and the three minimal pG-3-atoms (the numbers are the colors of the vertices and the surrounded vertices form the centers).

- *For all $i \in \{2, \ldots, t\}$, c_i has a neighbor in each of D_1, ..., D_{i-1}.*

The set $\{c_1, \ldots, c_t\}$ is called the center *of A and denoted by $C(A)$.*

Note that the sets D_1, ..., D_t induce a partial Grundy coloring of the pG-t-atom. Figure 2 illustrates several pG-t-atoms (and their induced colorings) obtained using the previous definition.

Observation 2.1. *For every pG-t-atom G, we have $|V(G)| \leq \frac{t(t+1)}{2}$.*

Lemma 2.2. *Let t and t' be two integers such that $1 \leq t' < t$. Every pG-t-atom contains a pG-t'-atom as induced subgraph.*

Proof. Every pG-t-atom G contains a pG-t'-atom G': we can obtain G' by removing every vertex in D_k, for $t' < k \leq t$, and by removing, afterwards, the vertices of G' not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$. □

Note that the only minimal pG-2-atom is P_2. The minimal pG-3-atoms are C_3, P_4 and $P_2 \cup P_3$. These graphs are illustrated in Figure 2.

Theorem 2.3. *For a graph G, we have $\partial\Gamma(G) \geq t$ if and only if G contains an induced minimal pG-t-atom.*

Proof. Suppose that $\partial\Gamma(G) = t'$ with $t' \geq t$. By definition, there exists a partial Grundy coloring of G with t' colors. Let u_1, ..., $u_{t'}$ be a set of Grundy vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a pG-t'-atom. Hence, by Lemma 2.2, since G contains an induced pG-t'-atom, then it also contains an induced minimal pG-t-atom.

Suppose G contains an induced minimal pG-t-atom. Thus, the sets D_1, ..., D_t induce a partial-Grundy coloring of this pG-t-atom. We can extend this coloring to a partial Grundy coloring of G with at least t colors in a greedy way by coloring the remaining vertices in any order, assigning to each of them the smallest color not used by its neighbors. □

Proposition 2.4. *Let G be a graph of order n and let t be an integer. There exists an algorithm in time $O(n^{\frac{t(t+1)}{2}})$ to determine if $\partial\Gamma(G) \geq t$. Hence, the problem pG-COL with parameter t is in XP.*

Proof. By Theorem 2.3, it suffices to verify that G contains an induced minimal pG-t-atom to have $\partial\Gamma(G) \geq t$. Since the order of a minimal pG-t-atom is bounded by $\frac{t(t+1)}{2}$, we obtain an algorithm in time $O(n^{\frac{t(t+1)}{2}})$. □

We finish this section by determining every graph G with $\partial\Gamma(G) = 2$.

Proposition 2.5. *For a graph G without isolated vertices, we have $\partial\Gamma(G) = 2$ if and only if $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or G only contains isolated edges.*

Proof. Zaker [23] has proven that $\Gamma(G) = 2$ if and only if G is the disjoint union of copies of some $K_{n,m}$, for $n \geq 1$ and $m \geq 1$. Let n and m be positive integers. We can note that a graph containing a copy of $K_{n,m}$, for $n \geq 2$ and $m \geq 1$ and a copy of $K_{n,m}$, for $n \geq 1$ and $m \geq 1$ contains an induced $P_3 \cup P_2$, hence a pG-3-atom. Hence, if $\partial\Gamma(G) = 2$, then $G = K_{n,m}$, for $n \geq 2$ and $m \geq 1$ or G only contains isolated edges.

Moreover, neither $K_{n,m}$ nor $P_2 \cup \ldots \cup P_2$ does contain an induced C_3, P_4 or $P_3 \cup P_2$. Hence, $\partial\Gamma(K_{n,m}) = 2$.

□

3 b-t-atoms: t-atoms for b-coloring

As in the previous section, we start this section with the definition of b-t-atoms (the notion of t-atom for b-coloring).

Definition 3.1. *Given an integer t, a b-t-atom is a graph A whose vertex-set can be partitioned into t sets $D_1, \ldots, D_t$, where D_i contains a special vertex c_i for each $i \in \{1, \ldots, t\}$ such that the following holds:*

- *For each $i \in \{1, \ldots, t\}$, D_i is an independent set and $|D_i| \leq t$;*
- *For all $i, j \in \{1, \ldots t\}$, with $i \neq j$, c_i has a neighbor in D_j.*

The set $\{c_1, \ldots, c_t\}$ is called the center of A and denoted by $C(A)$.

Note that the sets D_1, ..., D_t induce a b-coloring of the b-t-atom. Figure 3 illustrates several b-t-atoms (and their induced coloring) obtained using the previous definition.

Observation 3.1. *For every b-t-atom G, we have $|V(G)| \leq t^2$.*

Lemma 3.2. *Let t and t' be two integers such that $1 \leq t' < t$. Every b-t-atom contains a b-t'-atom as induced subgraph.*

Proof. Every b-t-atom G contains a b-t'-atom G': we can obtain G' by removing every vertex in D_k, for $t' < k \leq t$, and by removing, afterwards, the vertices not adjacent to any vertex in $\{c_1, \ldots, c_{t'}\}$. □

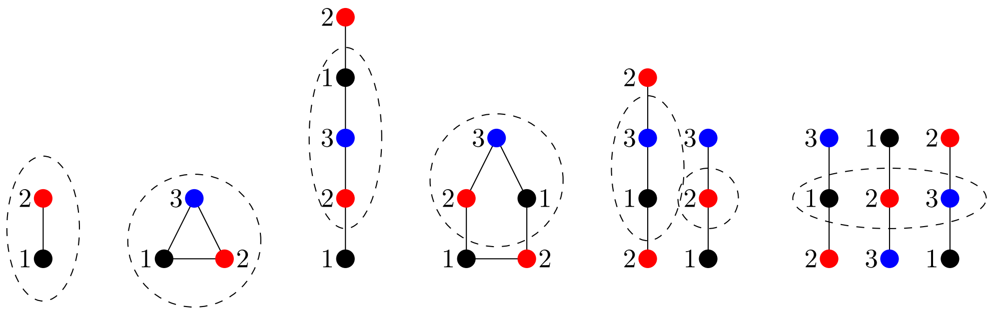

Figure 3: The minimal b-2-atom (on the left) and the five minimal b-3-atoms.

Note that the only minimal b-2-atom is P_2. The minimal b-3-atoms are C_3, P_5, C_5, $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$. These graphs are illustrated in Figure 3.

Observation 3.3. *Every minimal pG-t-atom is an induced subgraph of a minimal b-t-atom or a minimal t-atom (an atom for the Grundy number).*

Proposition 3.4. *Let G be a graph. If $\varphi(G) \geq t$, then G contains an induced minimal b-t-atom.*

Proof. Suppose that $\varphi(G) = t'$, with $t' \geq t$. Thus, there exists a b-coloring of G with t' colors. Let $u_1, \ldots, u_{t'}$ be a set of b-vertices, each in a different color class of $V(G)$. The graph induced by $N[u_1] \cup \ldots \cup N[u_{t'}]$ contains a b-t'-atom. Hence, by Lemma 3.2, since G contains an induced b-t'-atom, then it also contains an induced minimal b-t-atom. $\square$

Theorem 3.5. *For a graph G, we have $\varphi_r(G) \geq t$ if and only if G contains an induced minimal b-t-atom.*

Proof. Suppose that the graph G contains an induced b-t-atom A. Since A admits, by definition, a b-t-coloring, we have $\varphi_r(G) \geq t$. Using Proposition 3.4, we obtain the converse. $\square$

Definition 3.2. *Let G be a graph. For an induced subgraph A of G, let $N(A) = \{v \in V(G) \setminus V(A) |\ uv \in E(G),\ u \in V(A)\}$. A b-$t$-atom A is* feasible *in G if there exists a b-t-coloring of $V(A)$ that can be extended to the vertices of $N(A)$ without using new colors.*

Proposition 3.6. *Let G be a graph. If G contains an induced feasible minimal b-t-atom and no induced feasible minimal b-t'-atom, for $t' > t$, then $\varphi(G) = t$.*

Proof. Suppose that G contains an induced feasible minimal b-t-atom A and no b-t-coloring of G exists. We begin by considering that the vertices of $A \cup N(A)$ are already colored with t colors. We can note that, by assumption, no coloring of $A \cup N(A)$ (from the definition) can be extended to the whole graph using only t colors. Let t' be the largest integer such that the coloring can not be extended to a b-t'-coloring of the whole graph and let v be a vertex that can not be given a color among $\{1, \ldots, t'\}$. Thus, we suppose that the coloring can be extended to a b-$(t'+1)$-coloring where v is colored by $t'+1$. Since $A \cup N(A)$ is already colored, we have $v \in V(G) \setminus (A \cup N(A))$. The vertex v should be adjacent to vertices of every color, otherwise it could be colored. One vertex of each color class in $N(v)$ should be adjacent to vertices of each color class (except its color). Otherwise, the colors of the vertices of $N(v)$ could be changed in order that some color c no longer appear in $N(v)$, and consequently v can be recolored with color c. Then, the graph induced by the vertices at distance at most 2 from v contains a b-$(t'+1)$-atom where $N[v]$ contains the center of this b-$(t'+1)$-atom. Moreover, this b-$(t'+1)$-atom is feasible as the whole graph is b-$(t'+1)$-colorable, contradicting the hypothesis. □

Proposition 3.7. *Let G be a graph. If $\varphi(G) = t$, then G contains an induced feasible minimal b-t-atom and no induced feasible minimal b-t'-atom, for $t' > t$.*

Proof. Suppose $\varphi(G) = t$. By Proposition 3.4, G contains an induced minimal b-t-atom. If no induced minimal b-t-atom is feasible, then there exists no b-t-coloring of G, a contradiction. □

A direct consequence of Proposition 3.6 and Proposition 3.7 is the following.

Theorem 3.8. *For a graph G, we have $\varphi(G) = t$ if and only if G contains an induced feasible minimal b-t-atom and no induced feasible minimal b-t'-atom, for $t' > t$.*

The following proposition will be useful in the last section.

Proposition 3.9. *Let G be a graph and let $t = \varphi_r(G)$. If every minimal b-t-atom is feasible in G, then $\varphi(G) = \varphi_r(G)$.*

Proof. Since $t = \varphi_r(G)$, G does not contain a b-$(t+1)$-atom. Thus, by Proposition 3.6, we obtain $\varphi(G) = t$. □

Note that the problem of determining if a graph has a b-t-coloring is NP-complete even if t is fixed [21]. However, it does not imply that determining if $\varphi(G) \geq t$ for a graph G is NP-complete. In contrast with the b-chromatic number, determining if a graph has b-relaxed number at least t is in XP.

Proposition 3.10. *Let G be a graph of order n and let t be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi_r(G) \geq t$. In particular, the problem b-r-COL with parameter t is in XP.*

Proof. By Theorem 3.5, it suffices to verify that G contains an induced minimal b-t-atom to determine if $\varphi_r(G) \geq t$. By Observation 3.1, the order of a minimal b-t-atom is bounded by t^2. Thus, we obtain an algorithm in time $O(n^{t^2})$. □

Another NP-complete problem is to determine the *b-spectrum* of a graph G [2], i.e. the set of integers k such that G is b-k-colorable. For a graph G satisfying $\varphi(G) = \varphi_r(G)$, our algorithm can be used. Thus, proving that for a class of graphs, every graph G satisfies $\varphi(G) = \varphi_r(G)$, implies that the problem b-COL with parameter t is in XP for this class of graphs.

4 b-critical vertices and edges

The concept of *b-critical vertices* and *b-critical edges* has been introduced recently and since five years a large number of articles are considering this subject [1, 4, 5, 9, 24]. In this section, we illustrate how this notion is strongly connected with the concept of b-t-atom.

Definition 4.1 ([4, 9])**.** *Let G be a graph. A vertex v of G is b-critical if $\varphi(G-v) < \varphi(G)$. An edge e is b-critical if $\varphi(G-e) < \varphi(G)$. A vertex v (edge e, respectively) in a graph G is a* b-t-trap*, if there exists a b-t-atom of G that becomes feasible by removing v (e, respectively).*

Proposition 4.1. *Let G be a graph. A vertex v is b-critical if and only if it is in every feasible minimal b-$\varphi(G)$-atom and v is not a b-$\varphi(G)$-trap.*

Proof. Let $t = \varphi(G)$. First, if v is not in a feasible minimal b-t-atom, then $\varphi(G-v) = t$ and v is not b-critical. If v is a b-t-trap, then, by definition, $\varphi(G-v) = t$. Second, suppose v is not a b-t-trap. If v is in every feasible minimal b-t-atom, then, since every minimal b-t-atom in G does not contain any other feasible minimal b-t-atom as induced subgraph, $G-v$ does not contain a feasible minimal b-t-atom. Thus, v is b-critical. □

Corollary 4.2. *If a graph G contains two induced feasible minimal b-$\varphi(G)$-atoms with disjoint set of vertices, then it contains no b-critical vertex.*

Proposition 4.3. *Let G be a graph and v be a vertex of $V(G)$. If $\varphi(G-v) > \varphi(G)$, then G contains a minimal b-$\varphi(G-v)$-atom which is not feasible. If $\varphi(G-v) < \varphi(G) - 1$, then $G-v$ contains no feasible minimal b-t-atom, for $\varphi(G-v) < t \leq \varphi(G)$.*

Proof. Note that every b-t-atom contained in $G-v$ is also contained in G, for any integer t. Thus, if $\varphi(G-v) > \varphi(G)$, then G contains a b-$\varphi(G-v)$-trap and consequently a minimal b-$\varphi(G-v)$-atom which is not feasible. Moreover, if $\varphi(G-v) < \varphi(G) - 1$ and $G-v$ contains a feasible b-t-atom for $\varphi(G-v) < t \leq \varphi(G)$, then $\varphi(G-v) \geq t$. □

In [1], Balakrishnan and Raj have proved the following theorem.

Theorem 4.4 ([1])**.** *Let G be a graph and v be a vertex of $V(G)$. We have $\varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2 \leq \varphi(G-v) \leq \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$.*

Moreover, they have determined the families of graphs for which there exists a vertex v such that $\varphi(G-v) = \varphi(G) - \lfloor \frac{|V(G)|}{2} \rfloor + 2$ or $\varphi(G-v) = \varphi(G) + \lfloor \frac{|V(G)|}{2} \rfloor - 2$. In contrast with the b-chromatic number, we have the following property about the b-relaxed number.

Proposition 4.5. *Let G be a graph. If a vertex v is b-critical, then $\varphi_r(G-v) = \varphi_r(G) - 1$.*

Proof. By Proposition 4.1, v is in every b-$\varphi(G)$-atom. Let i be the integer associated to v in the construction of this b-$\varphi(G)$-atom. By removing the vertices with associated integer i, we obtain a b-$(\varphi(G)-1)$-atom and thus $\varphi_r(G-v)=\varphi_r(G)-1$. □

Note that this proposition was already proved for trees [4].

Lemma 4.6. *Let G be a graph with $4 \leq |V(G)| \leq 5$ and $E(G) \neq \emptyset$. We have $\varphi_r(G-v)=\varphi_r(G)+\lfloor\frac{|V(G)|}{2}\rfloor-2$, for every vertex v of $V(G)$, if and only if G contains two disjoint edges but no induced minimal b-3-atom.*

Proof. We can note that we have $\varphi_r(G-v)=\varphi_r(G)+\lfloor\frac{|V(G)|}{2}\rfloor-2$ if and only if $\varphi_r(G-v)=\varphi_r(G)$.

First, if G contains no minimal b-3-atom and contains an edge, then $\varphi_r(G)=2$. Moreover, if G contains two disjoint edges, then for any vertex v, $G-v$ contains P_2 and $\varphi_r(G-v)=2$.

Second, suppose that for every vertex v, $\varphi_r(G-v)=\varphi_r(G)$. The only minimal b-3-atoms that contains at most five vertices are K_3, C_5 and P_5. Moreover, the only minimal b-4-atoms and b-5-atoms that contain at most five vertices are K_4 and K_5. We are going to show that G is not one of these graphs

Case 1: $\varphi_r(G)=5$. If G is a K_5, then, by removing any vertex v, we obtain $\varphi_r(G-v)=4$.

Case 2: $\varphi_r(G)=4$. If G is a K_4, then, by removing any vertex v, we obtain $\varphi_r(G-v)=3$. If G contains an induced K_4, $|V(G)|=5$ and G is not K_5, then there exists a vertex v such $G-v$ has no induced K_4 and $\varphi_r(G-v)=3$.

Case 3: $\varphi_r(G)=3$. If G contains an induced K_3 and no induced K_4, then, since the induced K_3 in G have a common vertex v, we obtain $\varphi_r(G-v)=2$. Moreover, if G is P_5 or C_5, then, by removing any vertex v, we obtain $\varphi_r(G-v)=2$.

Thus, we can suppose that $\varphi_r(G)=2$. If G contains only edges with a common vertex v, then $\varphi_r(G-v)=1$. Hence, G contains no b-3-atom and contain two disjoint edges. □

The following theorem is a generalization of a conjecture of Blidia et al. [3] for the parameter φ_r. Note that the graphs P_4, C_4 and $P_2 \cup P_2$ do not contain any induced minimal b-3-atom and contain two disjoint edges.

Theorem 4.7. *Let G be a graph. We have $\varphi_r(G-v)=\varphi_r(G)+\lfloor\frac{|V(G)|}{2}\rfloor-2$, for every vertex v of $V(G)$, if and only one of these conditions is true about G:*

i) *G is P_2 or C_3.*

ii) *$E(G)=\emptyset$ and $4 \leq |V(G)| \leq 5$.*

iii) *$4 \leq |V(G)| \leq 5$ and G contains two disjoint edges but no b-3-atom.*

Proof. Note that if $|V(G)| \geq 6$, then, by Proposition 4.5, we can not have $\varphi_r(G-v) = \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$. Note also that if G contains only one vertex, then it can not satisfy $\varphi_r(G-v) = \varphi_r(G) + \lfloor |V(G)|/2 \rfloor - 2$.

First, if $2 \leq |V(G)| \leq 3$, then we have $\varphi_r(G-v) = \varphi_r(G) - 1$ if and only if G is a minimal b-t-atom. Hence, if and only if G is P_2 or C_3. Second, if G contains no edges, then $\varphi_r(G) = 1$ and for any vertex v, $\varphi_r(G-v) = 1$. The third condition is obtained by Lemma 4.6. □

Definition 4.2. *Let t be a positive integer and A be a b-t-atom. An edge e is* b-atom-critical *in A if $A - e$ is not a b-t-atom.*

Proposition 4.8. *Let G be a graph. An edge e is b-critical if and only if it is b-atom-critical in every feasible minimal b-$\varphi(G)$-atom and e is not a b-$\varphi(G)$-trap.*

Proof. Let $t = \varphi(G)$. First, if e is not b-atom-critical in a feasible minimal b-t-atom, then $G - e$ contains a feasible minimal b-t-atom and $\varphi(G-e) = t$. If e is a b-t-trap, then, by definition, $\varphi(G-e) = t$. Second, suppose that e is not a b-t-trap. If e is b-atom-critical in every feasible minimal b-t-atom, then, since every feasible minimal b-t-atom in G does not contain any other feasible minimal b-t-atom as subgraph in $G - e$, the graph $G - e$ does not contain a feasible minimal b-t-atom. Thus, e is b-critical. □

Corollary 4.9. *If a graph G contains two induced feasible minimal b-$\varphi(G)$-atoms with disjoint sets of b-atom-critical edges, then G contains no b-critical edge.*

5 b-perfect graphs

A b-perfect graph is a graph for which every induced subgraph satisfies that its b-chromatic number is equal to its chromatic number. More generally, we present the following definitions.

Definition 5.1 ([13]). *A graph G is* b-χ-k-bounded*, for k a positive integer, if $\varphi(G') - \chi(G') \leq k$, for every induced subgraph G' of G. A graph G is a* χ-k-unbounded b-atom*, for k a positive integer, if $\varphi(G) - \chi(G) > k$ and G is a b-t-atom for some integer t. A graph G is an* imperfect b-atom*, for k a positive integer, if $\varphi(G) > \chi(G)$ and G is a b-t-atom for some integer t.*

Hoang et al. [14] characterized b-perfect graphs by giving the family $\mathcal{F}$ of forbidden induced subgraphs depicted in Figure 4. We recall the following theorem:

Theorem 5.1 ([14]). *A graph is b-perfect if and only if it contains no graph from $\mathcal{F}$ as induced subgraph.*

Note that every graph in the family $\mathcal{F}$ is a b-t-atom for some t. More precisely, F_1, F_2 and F_3 are the only minimal bipartite b-3-atoms. The remaining

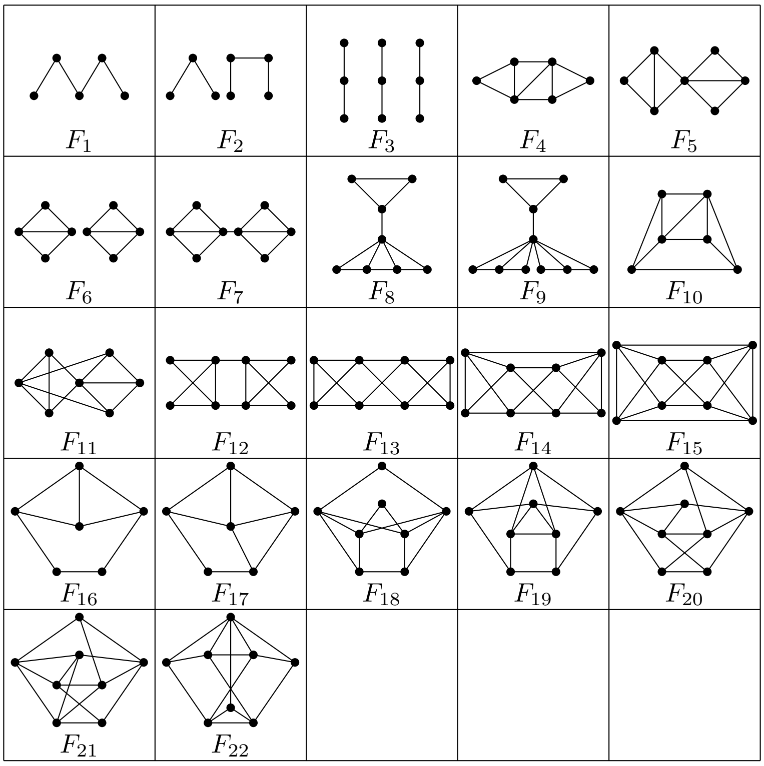

Figure 4: The family $\mathcal{F}$: the imperfect b-atoms [14].

graphs are minimal b-4-atoms that do not contain F_1, F_2 and F_3 as induced subgraph and which admit a proper coloring with three colors (as mentioned in [15]). We can state the following property about b-t-atoms.

Theorem 5.2. *Let k be a positive integer. A graph G is not b-χ-k-bounded if and only if it contains a minimal χ-k-unbounded b-atom.*

Proof. First, if G contains a minimal χ-k-unbounded b-atom, then, by definition, G is not χ-k-bounded.

Second, suppose G is not b-χ-k-bounded. Then, there exists an induced subgraph A of G of minimal order which is not b-χ-k-bounded. By removing vertices of A we can only decrease the chromatic number. Thus, by removing vertices we can obtain a b-$\varphi(A)$-atom which is χ-k-unbounded. $\square$

Corollary 5.3. *The graphs with b-chromatic number t which are b-χ-k-bounded,, for fixed integers k and t, can be defined by forbidding a finite family of induced subgraphs: the χ-k-unbounded b-atoms. Hence, a graph G is b-perfect if and only if it does not contain imperfect b-atoms.*

Let b-χ-BOUNDED be the following decision problem and let k be an integer, with $0 \leq k < \varphi(G)$.

> **b-χ-k-BOUNDED**
> **Instance** : A graph G.
> **Question**: Does $\varphi(G) - \chi(G) \geq k$?

By Corollary 5.3, we obtain the following corollary:

Corollary 5.4. *Let G be a graph and k be an integer, with $0 \leq k < \varphi(G)$. There exists an algorithm in time $O(n^{\varphi(G)^2})$ to solve b-χ-k-BOUNDED.*

Since a graph G is b-perfect if and only if it does not contain imperfect b-atoms, we have the following theorem:

Theorem 5.5. *The number of imperfect b-atoms is finite. A graph is an imperfect b-atom if and only if it is in the family $\mathcal{F}$ (Figure 4).*

The previous theorem is a consequence of Theorem 5.1. Remark that if we can prove that every minimal b-4-atom except K_4 contains an induced subgraph of the family $\mathcal{F}$, then, using Theorem 5.2, we obtain another proof of Theorem 5.1.

6 b-chromatic and b-relaxed chromatic numbers

In this section we consider the b-relaxed number relatively to the b-chromatic number and prove equality for trees and graphs of girth ar least 7.

Lemma 6.1. *A minimal b-t-atom has at most t connected components.*

Proof. Suppose that a minimal b-t-atom G has more than t connected component. By definition, at least one connected component A of G does not contain a vertex of $C(G)$. Since $G - A$ is also a b-t-atom, G is not minimal. □

Note that a minimal b-t-atom G contains a center $C(G)$ and the remaining vertices of G are neighbors of vertices of $C(G)$.

Proposition 6.2. *For a tree T, we have $\varphi(T) = \varphi_r(T)$.*

Proof. Let $t = \varphi_r(T)$. By Proposition 3.9, it suffices to prove that every minimal b-t-atom is feasible to have $\varphi(T) = \varphi_r(T)$. Let T' be a minimal b-t-atom and let $N[T'] = V(T') \cup N(T')$. By Lemma 6.1, T' has at most t connected components. Let u be a vertex of $N(T')$ with a maximal number of neighbors in $N[T']$. Since T' has at most t connected components and T is a tree, u has at most t neighbors in $N[T']$.

Our proof consists in extending the coloring of T' induced by $D_1, \ldots, D_t$ to $N(T')$ using colors from $\{1, \ldots t\}$. For $t = 2$, the proof is trivial since the only minimal b-2-atom is P_2 and we can easily extend the coloring to $N(P_2)$. Thus we can suppose that $t \geq 3$. If u has at most $t - 1$ neighbors in $N[T']$, then we can extend the coloring. Thus, we suppose that u has t neighbors in $N[T']$. In this case, T' has t connected components which are all stars. Each vertex of $N(u) \cap N[T']$ is either a vertex of a connected component of T' or a vertex in $N(T')$ which is adjacent to one vertex of $V(T')$. In these two cases the vertices of $N(u) \cap N[T']$ should be in or be adjacent to vertices of disjoint connected components of T'. Thus the vertices of $N(u) \cap N(T')$ have at most two neighbors in $N[T']$: the vertex u and another vertex of T' (otherwise, there is a cycle in T) . We begin by giving a color from $\{1, \ldots, t\}$ to the vertices of $N(T') \setminus \{u\}$. The vertex u can not be adjacent to all vertices of $C(T')$ since otherwise it would contradict $t = \varphi_r(T)$. Let $v \in N[T'] \setminus C(T')$ be a neighbor of u. If $v \in N(T')$, then v has at most two neighbors in $N[T']$ and v can be recolored in order to color u. If all neighbors of u are in T', then $v \in N(c_i)$, for $i \in \{1, \ldots, t\}$ and we can exchange the color of v with the color of a vertex $w \in N(c_i) \setminus \{v\}$ in order to color u (since $t \geq 3$, $N(c_i) \setminus \{v\}$ is not empty). Finally, the vertices of $N(w) \cap N(T')$ can be recolored if we have obtained an improper coloring by recoloring w. □

The *girth* of a graph G is the length of a smallest cycle in G. We finish this paper by proving that when a graph G has sufficiently large girth, we have $\varphi(G) = \varphi_r(G)$, thus extending Proposition 6.2.

Theorem 6.3. *Let G be a graph with girth g and $\varphi_r(G) \geq 3$. If $g \geq 7$, then $\varphi(G) = \varphi_r(G)$.*

Proof. Let $t = \varphi_r(G)$. By Proposition 3.9, it suffices to prove that every minimal b-t-atom is feasible to have $\varphi(G) = \varphi_r(G)$. Let A_t be a minimal b-t-atom. Our proof consists in extending the coloring of A_t induced by $D_1, \ldots, D_t$ to $N(A_t)$ using colors from $\{1, \ldots t\}$. Thus, we consider that the vertices of A_t are already colored.

For a vertex $u \in N(A_t)$, we denote by $I_c(u)$ the set $\{i \in \{1, \ldots, t\} | \; \exists v \in N(u) \cap N[c_i]\}$. For a vertex $u \in V(A_t)$, we denote by c^u a neighbor of u in $C(A_t)$ if $u \notin C(A_t)$ or the vertex u itself if $u \in C(A_t)$. Finally, we denote by $N[A_t]$, the set of vertices $V(A_t) \cup N(A_t)$. In the different cases, when we describe a cycle of length at most k by u_1-…-u_k, it is assumed that, depending the configuration, consecutive symbols can denote the same vertex. In this proof, any considered vertex is supposed to be in $N[A_t]$. We begin by proving the following properties:

i) No vertex of $N(A_t)$ is adjacent to two vertices of $N[c_i]$, for $1 \leq i \leq t$;

ii) If $u, v \in N(A_t)$ and $i \in I_c(u) \cap I_c(v)$, then u and v are not adjacent and have no common neighbor in $N(A_t) - c_i$;

iii) If $u, v \in N[c_i]$ and $u', v' \in N[c_j]$, $u \neq v$, $u' \neq v'$, for some i and j, $1 \leq i < j \leq t$, then the subgraph induced by $\{u, v, u', v'\}$ contains at most one edge.

i) If u is adjacent to two vertices of $N[c_i]$, for some i, $1 \leq i \leq t$, then u is in a cycle of length at most 4. This cycle contains u, c_i and one or two vertices of $N[c_i]$.

ii) If u and v are adjacent or have a common neighbor, then u and v belong to a cycle of length at most 6. This cycle contains u, v, vertices of $N[c_i]$ and possibly the common neighbor of u and v in $N(A_t) - c_i$, for i an integer such that $i \in I_c(u) \cap I_c(v)$.

iii) If the subgraph induced by $\{u, v, u', v'\}$ contains at least 2 edges, then there is a cycle of length at most 6 in G. This cycle is u-v-c_i if u and v are adjacent, u'-v'-c_j if u' and v' are adjacent or the cycle u-c_i-v-u'-v'-c_j, otherwise.

We are going to prove that either each vertex $u \in N(A_t)$ can be colored with colors from $\{1, \dots, t\}$ or the graph G contains a b-$(t+1)$-atom (which contradicts $\varphi_r(G) = t$). By properties i) and ii), any vertex of $N(A_t)$ has at most t neighbors in $N[A_t]$. Hence we may suppose that any vertex $u \in N(A_t)$ with less than t neighbors in $N[A_t]$ is already colored and only consider vertices of $N(A_t)$ with t neighbors in $N[A_t]$. For a vertex $u \in N[A_t]$, a color i is said to be *available* for u if no vertex has color i in $N(u) \cap N[A_t]$ (and therefore, u has no available color if the colors $1, \dots, t$ are not available for u). Let $N_*(A_t)$ be the set of vertices in $N(A_t)$ with no available colors.

We define the following three sets:

- $N_1 = \{u \in N(A_t) |\ N(u) \cap (V(A_t) \setminus C(A_t)) \neq \emptyset,\ N(u) \cap N(A_t) = \emptyset\}$;
- $N_2 = \{u \in N(A_t) |\ N(u) \cap (V(A_t) \setminus C(A_t)) \neq \emptyset,\ N(u) \cap N(A_t) \neq \emptyset\}$;
- $N_3 = \{u \in N(A_t) |\ N(u) \cap (V(A_t) \setminus C(A_t)) = \emptyset\}$.

We can remark that $N_1 \cup N_2 \cup N_3 = N(A_t)$.

In the remainder of the proof we will first consider the vertices of N_1; secondly the vertices of N_2; and finally the vertices of N_3.

Case 1: vertices of N_1.

Let u be a vertex of N_1. We recall that, by the above assumption, u has exactly t neighbors in A_t. Moreover, by Property i), $|I_c(u)| = t$. Let $c_i \in C(A_t)$. We denote by A^i_* the vertices of $N(c_i)$ which have a neighbor in $N_*(A_t)$. Notice that a vertex $v \in A^i_*$ can not have a neighbor x in $V(A_t) \setminus \{c_i\}$ since otherwise it would create a cycle v-x-c^x-v'-u, for u the neighbor of v in $N_1 \cap N_*(A_t)$ and v' the neighbor of u in $N[c^x]$. This cycle has length at most 5, contradicting $g \geq 7$. If for a vertex $c_i \in C(A_t)$ we have $|A^i_*| \geq 2$, we exchange the colors of the vertices of A^i_* by doing a cyclic permutation of their colors. Afterwards, we obtain that some vertices of $N_1 \cap N_*(A_t)$ have now an available color and we recolor them by any available color. Finally, we color the vertices of N_1, when possible, by any available color. Let $N_{**}(A_t)$ be the set of the remaining uncolored vertices of N_1. In the following subcases, we recolor at most once the vertices of $N[c_i]$, for $i \in \{1, \dots t\}$, since any two vertices of $N_{**}(A_t)$ can not both have neighbors in $N(c_i)$.

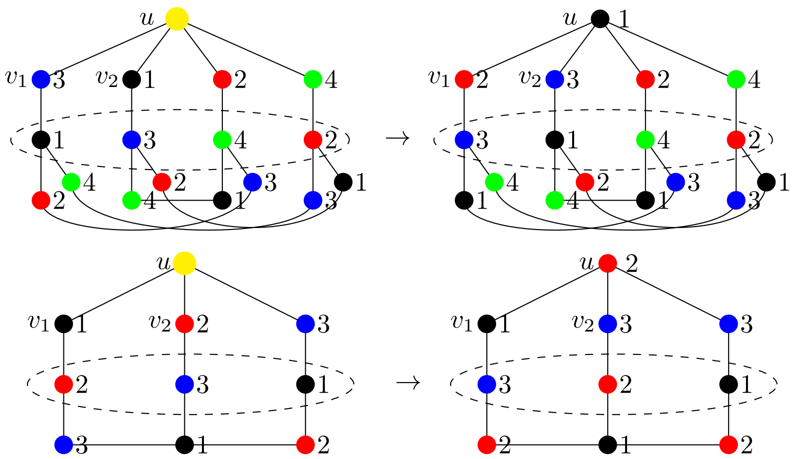

Figure 5: Possible configurations in Subcases 1.2.1 (on the top) and 1.2.2 (on the bottom) before (on the left) and after (on the right) the recoloring process.

By considering that $N_{**}(A_t) \neq \emptyset$ (or else we have nothing more to do in Case 1)), we can suppose that for every two integers i, j, $1 \leq i \neq j \leq t$, we have $N[c_i] \cap N[c_j] = \emptyset$. Otherwise, if there exists a vertex $u \in N_{**}(A_t)$ and a vertex $w \in N[c_i] \cap N[c_j]$, there is a cycle u-v-c_i-w-c_j-v' of length at most 6, for v a neighbor of u in $N[c_i]$ and v' a neighbor of u in $N[c_j]$. Thus, we obtain that if $N_{**}(A_t) \neq \emptyset$, then every vertex $c_i \in C(A_t)$ has only one neighbor of color j, for $1 \leq i \neq j \leq t$, since otherwise it would contradict the minimality of A_t (by removing one vertex of color j).

We then consider the two following subcases, for $u \in N_{**}(A_t)$.

Subcase 1.1: u has exactly one neighbor in $V(A_t) \setminus C(A_t)$.
Let v' be the neighbor of u in $V(A_t) \setminus C(A_t)$ and let c' be the color of v'. Notice that no vertex x from $N[c^{v'}]$ has a neighbor y in $V(A_t) \setminus N[c^{v'}]$, since otherwise it would create a cycle u-v'-$c^{v'}$-x-y-c^y of length at most 6. Consequently, we can exchange the color of v' with the color of one vertex from $N(c^{v'})$ and color u by c'.

Subcase 1.2: u has more than one neighbor in $V(A_t) \setminus C(A_t)$.
Let v_1 and v_2 be two neighbors of u in $V(A_t) \setminus C(A_t)$. Let c' be the color of v_1 and let c'' be the color of v_2.

If v_1 has a neighbor $x \in V(A_t) \setminus N[c^{v_1}]$, then there exists a cycle u-v_1-x-c^x-v' in G, with v' a neighbor of c^x in $N(u)$ (in the case c^x is not a neighbor of u). Similarly if v_2 has a neighbor in $V(A_t) \setminus N[c^{v_2}]$, then there is a cycle of length at most 5 in G. Consequently, we can suppose that v_1 has no neighbor in $V(A_t) \setminus N[c^{v_1}]$ and that v_2 has no neighbor in $V(A_t) \setminus N[c^{v_2}]$. If there exists a vertex of $N(c^{v_1}) \setminus \{v_1\}$ with no neighbor of color c', then we exchange the color of v_1 with the color of this vertex and color u by c'. If there exists a vertex of $N(c^{v_2}) \setminus \{v_2\}$ with no neighbor of color c'', then we exchange the color of v_2 with the color of this vertex and color u by c''. Thus, we may suppose that every vertex w of $N(c^{v_1}) \setminus \{v_1\}$ (of $N(c^{v_2}) \setminus \{v_2\}$, respectively) has a neighbor $\overline{w}$ of color c' (c'', respectively) in $V(A_t)$. We consider three subscases in order to color to u.

Subcase 1.2.1: the vertices v_1 and c^{v_2} have the same color and the vertices v_2 and c^{v_1} have the same color.
Notice that no vertex $w \in N(c^{v_1})$ is adjacent to c^{v_2} since otherwise u-v_1-c^{v_1}-w-c^{v_2}-v_2 would be a cycle of length at most 6 in G. For the same reason, no vertex $w \in N(c^{v_2})$ is adjacent to c^{v_1}. Thus, by Property iii), no vertex $w \in N[c^{v_1}] \cup N[c^{v_2}]$ has a neighbor $x \in V(A_t) \setminus (N[c^{v_1}] \cup N[c^{v_2}] \cup \{\overline{w}\})$, since there exists a vertex $y \in N(c^w)$ with neighbor $\overline{y} \in N(c^x)$. There could exist two adjacent vertices w and w' with $w \in N(c^{v_1})$ and $w' \in N(c^{v_2})$. However, the vertex w' has no neighbor of color c'' in A_t since w' and v_2 can not be adjacent and there does not exist a second vertex of color c'' in $N(c^{v_2})$. Consequently, we can exchange the color of v_1 with the color of v_2, the color of c^{v_1} with the color of c^{v_2} and afterward we can exchange the color of one vertex from $N(c^{v_1}) \setminus \{v_1\}$ with the color of v_1 and color u by c''. The top of Figure 5 illustrates this recoloring process on a minimal b-4-atom fulfilling the hypothesis of Subcase 1.2.1.

Subcase 1.2.2: the vertices v_1 and c^{v_2} do not have the same color.
Let i be the color of c^{v_1} and j be the color of c^{v_2}. In this case, we exchange the color of c^{v_1} with the color of c^{v_2} and the color of the vertex w of color j in $N(c^{v_1})$ with the color of the vertex w' of color i in $N(c^{v_2})$. For this, we have to suppose that w is not adjacent to a vertex of color i and that w' is not adjacent to a vertex of color j. For $t \geq 4$, such vertices w and w' exist since at most one vertex of $N(c^{v_1})$ has a neighbor of color j (otherwise, it would contradict Property iii) since every vertex of $N(c^{v_1}) \setminus \{v_1\}$ has already a neighbor in $V(A_t)$ of color c') and at most one vertex of $N(c^{v_2})$ has a neighbor of color i. If $t = 3$, then the only (up to isomorphism) b-3-atom with a coloring fulfilling all these hypothesis (up to color permutation) is illustrated at the bottom of Figure 5, along with the recoloring process. In this b-3-atom, no more edge can be added (otherwise, it would create a cycle of length at most 6).

Subcase 1.2.3: the vertices v_2 and c^{v_1} do not have the same color.
We proceed as for the previous subcase by considering v_2 instead of v_1 and c^{v_1} instead of c^{v_2}.

Case 2: vertices of N_2.
Since each pair of adjacent vertices $u, v \in N(A_t)$ satisfies Property ii), we obtain that $I_c(u) \cap I_c(v) = \emptyset$. We color each vertex $u \in N_2$ by a color $i \in I_c(u)$ such that u and c_i are not adjacent.

Case 3: vertices of N_3.

Notice that, by definition, a vertex of $C(A_t)$ has no available color. Let $u \in N_3$. We begin by coloring u with any available color if it has some. If u has no available color, there could exist a color i such that every vertex of $N(u)$ with color i has an available color (these vertices should be in $N(A_t)$). If such color i exists, we recolor these vertices of color i by any available color and give color i to u. If such color i does not exist, then the set of vertices at distance at most 2 from u induces a b-$(t+1)$-atom with center $N[u]$. It can be noted that the recolored vertices are in $N(A_t)$ since $N(u) \cap V(A_t) \subseteq C(A_t)$.

We finish this proof by illustrating that the obtained coloring is a b-t-coloring of $N[A_t]$. In case 1, we have modified the coloring of A_t. However, since we have exchanged the colors of well-chosen vertices in order that every vertex of $C(A_t)$ still has neighbor of every color from $\{1, \dots, t\}$ except its own color, this coloring remains a b-t-coloring. In case 3, we have only changed the color of vertices from $N(A_t)$.

□

We think that the previous theorem can be useful to determine the family of graphs of girth at least 7 satisfying $\varphi(G) = m(G)$. It has already been proven that graphs of girth at least 7 have b-chromatic number at least $m(G) - 1$ [7].

Corollary 6.4. *Let G be a graph of girth at least 7 and of order n and let t be an integer. There exists an algorithm in time $O(n^{t^2})$ to determine if $\varphi(G) \geq t$.*

7 Open questions

We conclude this article by listing few open questions.

1. For which family of graphs are the b-relaxed number and the b-chromatic number equal?

2. Does there exist an easy characterization of feasible b-t-atoms?

3. Does there exist an FPT algorithm, with parameter t, to determine if $\varphi(G) \geq t$?

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